On the Forms of Unicursal Sextic Scrolls.

By Virgil Snyder.

The study and classification of scrolls whose order is less than or equal to 5 has been, to quite an extent, completed. The following paper will discuss some types of sextic scrolls.

The literature on cubic and quartic scrolls is extensive and well known. The only paper on quintics with which I am familiar is that by H. A. Schwarz, "Ueber die geradlinigen Flächen fünften Grades," in Crelle's Journal, Vol. 67, pp. 23–57. Apart from incidental mention of particular cases, there has been no investigation of sextics. I have used the methods of Cayley, Salmon, and Schwarz, but the new singularities which first appear in the sextic sometimes require a different treatment.

The first section is a direct generalization of the corresponding articles in Schwarz's paper, and I have used his notation throughout; the remaining treatment differs widely in the details from his procedure. In the second section, Schwarz's method of generation is employed in part, and in the third section, the dual of his method.

With the exception of the generators, all plane curves which lie upon a scroll which is not reducible, and which possess the property that only one generator passes through each point, must be of the same genus. This follows from the fact that the generators furnish a one to one correspondence between the points of the two curves.

This principle furnishes a basis of classifying scrolls according to the genus of the curves which can be drawn upon them.

When a straight line which is not a generator, a conic section, a nodal cubic or any unicursal curve lies on a scroll as a simple curve, then every section of the scroll will be unicursal, and the surface itself will be called unicursal, or $p = 0$. Similarly for $p = 1, \ldots.$

For the geometric construction of the various cases, it will be desirable to consider the surface as generated by joining corresponding points of two plane curves of the same genus, or, dually, by the line of intersection of corresponding planes of two developables.

§1.—*General Discussion of the Various Possible Types of Sextic Scrolls.*

A plane passed through any generator g of a sextic scroll will cut from the surface a curve of the fifth order. This curve cuts the generator g in five points. One of these is the point of tangency of the plane through g. In general, each plane is a simple tangent plane, hence the other points are double points on the surface, so that each generator of the scroll is cut by four other generators. There are, therefore, an infinity of planes which pass through two generators. The consideration of these planes furnishes the point of departure of the following discussion.

Every plane through two generators cuts from the surface a quartic curve, which may be reducible. Whenever a plane section of a scroll is reducible, the composite curve of section must consist of a number of generators and an irreducible curve which possesses the property that at least one generator passes through every point, if the surface is not a cone.

Cones will be excluded from the investigation. In case of the sextic scroll the irreducible curve in a plane containing two generators may be a simple or multiple straight line, a conic, a cubic or a quartic curve.

A. If the irreducible curve be a simple straight line, a conic, a nodal cubic, or a trinodal quartic, the surface is of genus 0.

Any scroll whose order is greater than 4, cannot contain more than one simple conic.

B. If the curve of section be a non-singular cubic or a binodal quartic, then $p = 1$.

C. When the curve is a quartic with one node, $p = 2$.

D. When the curve is a quartic without nodes, $p = 3$.

E. It is necessary to make a particular investigation of the case in which the surface contains a multiple linear directrix.

(a.) Suppose the directrix be double. Then every plane through the double line will cut four generators from the surface, because the section of the surface

made by this plane cannot contain any further irreducible part than the double line. Through every point of the double line passes one generator (in which case the directrix is itself a generator) or two. In the first case, the scroll is unicursal. In the second case, the plane which contains the two generators which pass through a point of the double line, does not contain the double line itself. Suppose this were the case. Then every plane through the double line would contain two generators which cut each other on the double line, and two other generators, neither of which can pass through the point of intersection of the first two, because the line was only double.

Hence each of the latter generators must cut the directrix in some other point. If they intersect in different points, then through each such point on the directrix d must pass one other generator of the surface which cannot lie in the plane of the first ones, as the complete section of the surface is accounted for, hence the plane through the last two does not contain d.

The only remaining alternative is that the two latter generators cut d in the same point. This configuration is excluded, as it would not give an irreducible sextic scroll.

Hence, in general, the plane of the two generators does not contain d. Consider the section of the surface made by the plane of the two generators issuing from the same point of d. This plane can contain no other generator, for every generator must cut the double line, hence it would have to pass through the point of intersection of the first two, making the directrix triple, contrary to hypothesis. The curve of section is, therefore, either an irreducible quartic curve or a fourfold line.

In the first case, p cannot be greater than 3. In the latter case, in which the surface has one double and one quadruple directrix, skew to each other, if a plane be passed through any generator, not containing either directrix, it will cut from the surface a quintic curve which has a triple point at the intersection of the generator and the quadruple directrix. If the surface does not contain any double generators, the quintic curve will have no other singular points, hence the curve, and therefore the surface, will have $p = 3$.

For each double generator, the genus of the surface would be reduced by one, hence a (2, 4) scroll may have 3 double generators without becoming reducible.

Every sextic scroll which contains a double generator contains an infinite

number of plane quartic curves whose planes all pass through the double generator.

When the surface is of type (2, 4), these plane curves must all have a double point at the intersection of the double generator and the quadruple directrix—each would then be of genus $p = 2$.

(b.) Suppose the plane of the two generators cuts from the surface a triple line. It will also contain a generator, then every plane through the triple line must cut from the surface three generators, as no other irreducible curve is possible, apart from the multiple directrix. Of these three generators, two may coincide with the directrix, or only one, or none. Being a triple line, three generators will issue from every point. If two of them coincide with the directrix, the triple line counts for simple directrix and double generator, hence the surface is unicursal.

Let two generators issue from each point of the triple line apart from the line itself. There are now two cases to consider, according as the plane of the two generators does or does not contain the triple line. In the first case the plane through the generators would cut from the surface just one more line. This line would lie in the plane π and cut d; through the point in which this extra generator cuts d must pass another generator not lying in the plane, hence the plane of the two latter generators would not contain d. The case, then, in which the two generators lie in a plane with d, can be excluded. The only remaining case is when the plane through the generators which issue from the same point of d does not contain d. The plane will then cut from the surface an irreducible quartic or a cubic and a generator or a conic for particular positions, or, finally, a multiple line, which would have to be a triple line. In this case, d could not be a generator, but 3 generators would lie in the same plane.

The case in which three generators issue from each point of d and all lie in the same plane not containing d, leads also to a new triple line. Every generator must intersect both skew triple directrices. A plane passed through any generator will cut from the scroll a quintic curve having a double point on each directrix and cutting the generator in one more point, the point of contact of the scroll.

The quintic curve has in general no further double points, hence it belongs to the type $p = 4$. As before, the type is reduced by a unit for every double generator, hence a (3, 3) scroll may have 0, 1, 2, 3, or 4 double generators.

(c.) When the irreducible curve contained in a plane passing through two generators is a fourfold line d, four different cases may arise.

From every point of d passes 1, 2, 3, or 4 generators of the surface which are different from d itself.

If only one generator (distinct from d) passes through each point of d, the surface is of genus $p = 0$.

Suppose two generators, distinct from d, pass through every point of d, and let their plane not contain d. This plane will cut from the surface an irreducible quartic, or a multiple directrix.

If the plane of the two generators always passes through the quadruple line, so that the two generators which any plane through d cuts from the surface intersect on d, then, apart from possible multiple generators, no other double lines exist on the surface. This is the (2, 4) case in which the two skew directrices coincide.

If a plane be passed through a generator of a scroll of this type, the section made by it will be a quintic curve having a triple point at the intersection of the generator and the quadruple directrix, and no other multiple point, hence the surface will belong to the category $p = 3$.

Suppose the fourfold line such that three generators, distinct from the line itself, pass through every point of it. Then a plane exists which contains two of them without passing through the fourfold line. This plane will then cut from the surface an irreducible quartic curve or a multiple line; the line is at most a double line.

Suppose, finally, that four generators distinct from the multiple directrix issue from each of its points. If all four lines lie in one plane, the surface is of type (2, 4) and the plane cuts the surface in a double line which is not a generator. This case has already been considered.

(d.) Suppose, finally, that the surface has a fivefold line. Then the plane of two generators contains a quartic curve which has a triple point on the fivefold directrix, hence the surface is unicursal.

It thus appears that sextic scrolls are to be divided into five groups, according as the genus is 0, 1, 2, 3, 4.

§2.—*Unicursal Sextic Scrolls Generated by Two Developables.*

When one variable has been eliminated, the equations of the variable line which generates the scroll is defined as that of intersection of the two planes

$$\left.\begin{aligned} E &\equiv at^m + bt^{m-1} + \ldots\ldots pt + q = 0, \\ E'' &\equiv a't^n + b't^{n-1} + \ldots\ldots p't + q' = 0, \end{aligned}\right\} m \geqq n,$$

in which a, b .... ; a', b' .... are linear homogeneous expressions in x, y, z, w. By eliminating t between these two equations, there results an equation in x, y, z, w of degree $m + n$, which is the order of the surface.

When the two equations $E = 0$, $E' = 0$ do not represent the same plane for any value of t, the eliminant will contain no extraneous factors. Suppose that $E \equiv E'$ when $t = t_0$. Then this plane appears as factor in the scroll, Consider, in this case, a constant $\varkappa$ so determined that $E - \varkappa E' \equiv 0$ when $t = t_0$, which is always possible. $E - \varkappa E' \equiv (t - t_0) E''$. Now, replace $E = 0$ by the plane $E'' = 0$. This process can be continued until no value of t exists for which E, $E^{(n)}$ define the same plane. For unicursal surfaces of the sixth order $m + n = 6$ hence there are three cases to consider:

$$\text{I.} \quad \begin{cases} m = 5 & E \equiv at^5 + bt^4 + ct^3 + dt^2 + et + f = 0, \\ n = 1 & E' \equiv pt \quad + q = 0. \end{cases}$$

$$\text{II.} \quad \begin{cases} m = 4 & E \equiv at^4 \; + bt^3 + ct^2 + dt + e = 0, \\ n = 2 & E' \equiv pt^2 \; + qt + r = 0. \end{cases}$$

$$\text{III.} \quad \begin{cases} m = 3 & E \equiv at^3 \; + bt^2 + ct + d = 0, \\ n = 3 & E' \equiv pt^3 + qt^2 + rt + s = 0. \end{cases}$$

Since the order of a scroll is not changed by duality, it is possible to think of it as generated by joining corresponding points of two twisted curves rather than by the lines of intersection of corresponding planes of two developables, hence

If a (1, 1) *correspondence exist between the points of a straight line and the points of a unicursal quintic, the lines joining corresponding points generate a sextic scroll.*

Similarly for a conic and unicursal quartic, and for two cubics.

[I.] $m = 5, n = 1$.

The equation of the scroll having a quintuple line can be written in the form

$$u_6 + zu_5 + wv_5 = 0, \tag{1}$$

in which u_s, v_s are binary quantics in x, y of order s. The five tangent planes at any point z_1, w_1 of the quintuple line are defined by the equation

$$z_1 u_5 + w_1 v_5 = 0.$$

For certain values of $z_1 : w_1$, 8 in number, two roots of this equation coincide they define the cuspidal generators, which cut the quintuple line in the pinch points. The pinch-points may also be defined as the points in which the quintuple line cuts the octic torse $a\lambda^5 + b\lambda^4 + \ldots\ldots = 0$.

By a suitable linear transformation, equation (1) may be reduced to

$$x^2y^2u_2 + zu_5 + wv_5 = 0.$$

This will be called type I.

A simple mode of generation is obtained from the section made on the scroll by the plane containing two generators. The section is a quartic curve having a triple point. The surface is generated by the lines cutting two such quartics not in the same plane, and the line joining the triple points. Various subforms exist, when the pinch-points coincide. When $z_1u_5 + w_1v_5 = 0$ has a cubic factor two pinch-points coincide, which requires that the octic torse touch the quintuple directrix.

When u_5, v_5 contain a common factor, so that the equation may be written

$$x^2y^2u_2 + (ax + by)(zu_4 + wv_4) = 0,$$

in one position of the generator it coincides with the directrix, $ax + by = 0$ being the osculating tangent plane. This is type II. Subforms exist as under I.

III. $x^2y^2u_2 + (ax + by)(cx + dy)(zu_3 + wv_3) = 0$, two such limiting generators and osculating planes.

IV. $x^2y^2u_2 + u_1 . v_1 . \bar{u}_1 (z\bar{u}_2 + wv_2) = 0$, three such generators.

V. Four such generators.

VI. $x^2y^2u_2 + u_1^2 (zu_3 + wv_2) = 0$. In this case two of the five sheets through the quintuple line unite in forming a single cuspidal edge.

VII. $x^2y^2u_2 + u_1^2 . v_1 (z\bar{u}_2 + wv_2) = 0$, cuspidal edge and limiting generator.

VIII. $x^2y^2u_2 + u_1^3 (z\bar{u} + wv_2) = 0$, three sheets in cuspidal edge, i. e. three sheets have common tangent plane.

IX. $x^2y^2u_2 + u_1^3 . v_1 (zu_2 + w\bar{v}_1) = 0$.
X. $x^2y^2u_2 + u_1^2v_1^2 (zu_1 + wv_1) = 0$.
XI. $x^2y^2u_2 + u_1^4 (z\bar{u}_1 + wv_1) = 0$.

In forms V, IX, X and XI, the quintuple line is also fourfold generator, so that only one other generator can issue from each point of the directrix. These scrolls belong to special linear congruences. Their asymptotic lines are algebraic and of order 9.

When in I, u_2 vanishes identically, the five generators which issue from the same point of the quintuple directrix lie in one plane, which also contains a simple linear directrix $z = 0$, $w = 0$, skew to the first one. This is type XII. The surface is contained in a general linear congruence ; its asymptotic lines are of order 10. The forms V, IX, X, XI may also be regarded as limiting cases of XII, when the two directrices approach coincidence.

Among the subforms of XII, two varieties are of particular interest, those which are transformed into themselves by a cyclic collineation of order 5 and those whose asymptotic lines are reducible. The scroll

$$zx\left(8x^2 - (3 - \sqrt{5})\, y^2\right)\left(32x^2 - 5\,(7 + 3\sqrt{5})\, y^2\right) = wy\left(8y^2 - (3 - \sqrt{5}x^2)\right)\left(32y^2 - 5\,(7 + 3\sqrt{5})\, x^2\right)$$

is of the first kind.*

In the scroll $$zx^5 + wy^5 = 0$$

there are two real fourfold pinch-points, at which all five generators coincide; from the other points of the multiple directrix issues but one real generator.

The surface $zx(3x^4 + 5y^4) + wy(5x^4 + 3y^4) = 0$ differs from the preceding by having four double pinch-points, all real, at each of which three generators coincide.

The surface $zx(3x^4 - 5y^4) + wy(5x^4 - 3y^4) = 0$ has four double pinch-points, all imaginary. From every point of the multiple directrix issue three real and distinct generators. The form $zx^3(x^2 - 5y^2) + wy^3(5x^2 - y^2) = 0$ has also three generators issuing from each point, with real double pinch-points. The form $zx^3(x^2 + 5y^2) + wy^3(5x^2 + y^2) = 0$ has two real and two imaginary double pinch-points. In all these forms, the asymptotic lines are of order 5. The necessary

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* Ameseder, in the Wiener Berichte for 1890, treats of this kind of cyclic collineation. It is a projection of an axial rotation through 72°.

condition is that at every pinch-point three generators coincide, so that all of the pinch-points coincide in pairs.*

[II.] $m = 4,\ n = 2.$

By writing

$$\begin{aligned}
&p\,[p^2d - r\,(bp - aq)] - q\,[p\,(cp - ar) - q\,(bq - aq)] \equiv \phi_1,\\
&p^3e - r\,[p\,(cp - ar) - q\,(bp - aq)] \equiv \phi_2,\\
&p^2qe - r\,[p^2d - r\,(bp - aq)] \equiv \phi_3,\\
&pq\,(qe - dr) + r\,[r\,(cp - ar) - p^2e] \equiv \phi_4,\\
&q\,[r\,(cr - ep) - q\,(dr - qe)] - r\,[r^2b - p\,(dr - qe)] \equiv \phi_5,
\end{aligned}$$

the results of partial elimination of t may be written

$$\phi_1 t + \phi_2 = 0,\quad \phi_2 t + \phi_3 = 0,\quad \phi_3 t + \phi_4 = 0,\quad \phi_4 t + \phi_5 = 0,$$

and the equation of the surface may be written in either of the following forms:

$$\begin{gathered}
\phi_2^2 - \phi_1\phi_3 = 0,\quad \phi_1\phi_4 - \phi_2\phi_3 = 0,\quad \phi_3^2 - \phi_2\phi_4 = 0,\\
\phi_1\phi_5 - \phi_2\phi_4 = 0,\quad \phi_2\phi_5 - \phi_3\phi_4 = 0,\quad \phi_4^2 - \phi_3\phi_5 = 0.
\end{gathered}$$

Each of these equations contains an extraneous factor; these are p^2, pq, pr, $q^2 - pr$, qr, r^2 respectively.

These expressions are not all independent. The relation

$$r\phi_1 - q\phi_2 + p\phi_3 \equiv 0$$

can be easily verified. By equating the values of t,

$$\frac{\phi_1}{\phi_2} = \frac{\phi_2}{\phi_3} = \frac{\phi_3}{\phi_4} = \frac{\phi_4}{\phi_5} = \frac{r\phi_1 - q\phi_2 + p\phi_3}{r\phi_2 - q\phi_3 + p\phi_4}, \tag{2}$$

hence, $$r\phi_2 - q\phi_3 + p\phi_4 \equiv 0,$$

and similarly, $$r\phi_3 - q\phi_4 + p\phi_5 \equiv 0.$$

The surfaces ϕ_1, ϕ_2 do not intersect in a plane curve, hence ϕ_3 passes through their curve of intersection; similarly for ϕ_4, ϕ_5, so that all the surfaces pass through the same curve. Again, from the form of equations (2), the non-reducible part of

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† A similar case of a (3, 1) scroll was discussed by me in the American Journal, Vol. 22, p. 257. The algebraic condition is discussed in Salmon's "Higher Algebra," 4th ed., p. 162. The Jacobian of u_5, v_5 must be a perfect square.

this curve is a double line on the sextic scroll. The surfaces ϕ_2, ϕ_3 have the four lines p, r; p, a; e, r; p, q in common; p, r is a double line on ϕ_3, and the two surfaces touch along the line p, a. The plane $r = 0$ touches ϕ_2 the whole length of p, r and is an inflexional plane. The total curve of intersection of ϕ_2, ϕ_3 is of order 16; the common lines account for a curve of order 6, hence the double curve on the scroll is of order 10. The point p, q, r is a threefold point on ϕ_2 and on ϕ_3, hence a ninefold point on their curve of intersection. The line p, r being a single line on ϕ_2 and a double line on ϕ_3, has a double point at p, r, q. Similarly, the line q, r has three points at the ninefold point; no other lines common to the two surfaces pass through the multiple point, hence the curve has a fourfold point.

The unicursal sextic scroll of form [*II*] *has a double curve of order* 10, *which has a fourfold point.*

This scroll will be called type XIII.

The surface $\phi_2^2 - \phi_1\phi_3 = 0$ is the envelope of the quadratic pencil of quartic surfaces

$$\phi_1 t^2 + 2\phi_2 t + \phi_3 = 0;$$

each surface of the pencil passes through the double curve; the residual curve is then of order 4. As the surface must always touch the scroll, the two must either touch along one generator and intersect in two others or touch along two.

The same may be said of the two other pencils,

$$\phi_2 t^2 + 2\phi_3 t + \phi_4 = 0,$$
$$\phi_3 t^2 + 2\phi_4 t + \phi_5 = 0.$$

Every generator cuts the double curve in 4 points, hence the curve cannot lie on a quadric or cubic scroll, for in that case every generator of the sextic would also belong to the simpler surface. Through every point of the double curve pass two generators, each of which cuts the double line in three other points. The cone of order 9 having the double curve for directrix and any point upon it for vertex, contains two triple and one fourfold generator, the latter passing through the fourfold points.

A non-reducible sextic scroll cannot have a double curve of order 10 with a double point of order higher than the fourth, for every line joining the multiple point to any other point of the curve would be a part of the surface, and this cone would be of order less than 6, hence the surface would be degraded.

This reasoning cannot be applied to cases in which the double curve of order 10 is itself reducible.

Some of the possible forms into which the double curve can break up will now be considered.

Let the surface have a fourfold directrix line. If $p_1 = 0$, $q_1 = 0$ be the equations of two planes through the line, then every other plane through the line can be represented by an equation of the form $p_1\lambda + q_1 = 0$.

Any plane through the directrix will cut two generators from the surface, which are generally distinct from the directrix. To every value of λ correspond therefore two values of t; to every value of t corresponds but one plane through the directrix, hence to one value of λ. λ is therefore a rational quadratic function of t.

By proper linear substitution this can always be brought to the form

$$\lambda = t^2, \quad \text{or} \quad pt^2 + q = 0,$$

which, associated with

$$at^4 + bt^3 + ct^2 + dt + e = 0,$$

defines the scroll. The equation of the surface is therefore

$$[p(qc - pe) - q^2a]^2 + pq(pd - bq)^2 = 0. \tag{3}$$

This method must be modified somewhat when one of the four generators which issue from each point of the multiple directrix coincides with the directrix itself In that case the equation $at^4 + bt^3 + ct^2 + dt + e = 0$ is expressible in the form

$$(a't^3 + b't^2 + c't + d')(t - t_0) + \alpha p + \beta q = 0.$$

Corresponding restrictions must be made when two or more generators coincide with the multiple directrix.

From equation (3), when $p = 0$, $q^4a^2 = 0$; when $q = 0$, $p^4e^2 = 0$, hence the planes p, q are both torsal. Besides the fourfold line p, q, the surface has as double curve the line of intersection of the cubic scroll $p(qc - pe) - q^2a = 0$ and the quadric $pd - bq = 0$. It is a quartic curve of the second kind; the generators of one system of the hyperboloid $pd - bq = 0$ meet it in three points, those in which they meet the cubic scroll. Those generators of the other system cut the curve but once, as they cut the double line which is a generator of the first system. The quartic cuts the fourfold line in three points. This is type XIV.

This quartic curve may now break up into a cubic and a straight line, into a

conic and two straight lines, or finally, into four straight lines. (A quartic of the second kind cannot be composed of two conics.)

When the quartic curve breaks up into a cubic and a straight line, it is necessary that the quadric has a second generator in common with the cubic scroll. Let this common generator be cut from the plane $p\lambda_0 + q = 0$. Then the plane $\lambda_0^2 a + \lambda_0 c + e = 0$ is identical with $\varkappa_0 (b\lambda_0 + d) + \mu_0 (p\lambda_0 + q) = 0$. Under the same hypothesis, the following identity also exists:

$$p(qc - pe) - q^2 a \equiv p\varkappa_0 (bq - pd) + (\lambda_0 ap + cp - bp\varkappa_0 - \mu_0 p^2 - qa)(p\lambda_0 + q),$$

which shows that the cubic curve lies on the quadrics $pd - bq = 0$, $\lambda_0 ap + cp - \varkappa_0 bp - \mu_0 p^2 - qa = 0$. It cuts the line p, q in two points. The double line $p\lambda_0 + q$, $b\lambda_0 + d$ cuts p, q in one point. The three lines together constitute the double curve of order 10. The cubic curve cuts the line $p\lambda_0 + q$, $b\lambda_0 + d$ in one point. If

$$\lambda_0 ap + cp - \varkappa_0 bp - \mu_0 p^2 - qa$$

be denoted by u, the equation of the surface becomes

$$\{\varkappa_0 p (bq - pd) + u (p\lambda_0 + q)\}^2 + pq (pd - bq)^2 = 0.$$

It is type XV. The second double line is a double generator of the surface.

The cubic curve may further break up into a conic and a straight line. In this case the two quadrics

$$u = 0, \quad qb - pd = 0$$

have a generator of each system in common; the second generator lies in the plane $p\lambda_0' + q = 0$; then

$$(\lambda_0 a + c - \varkappa_0 b - \mu_0 p) + \lambda_0' a \equiv \varkappa_0' (p\lambda_0' + q) + \mu_0' (b\lambda_0' + d),$$

which may be rearranged in the form

$$(\lambda_0 a + c - \varkappa_0 b - \mu_0 p) p - aq \equiv \mu_0' (pd - qb) - (a - \varkappa_0' p - \mu_0' b)(p\lambda_0' + q).$$

The second double line is $p\lambda_0' + q$, $b\lambda_0' + d$, and the conic is defined by

$$pd - qb = 0, \quad a - \varkappa_0' p - \mu_0' q = 0.$$

The configuration of multiple lines now consists of the fourfold line p, q; two double lines, each cutting p, q and skew to each other, and a conic which

cuts each of these three lines once. The equation of the surface is

$$\{[x_0 p + \mu_0' (p\lambda_0 + q)](pd - qb) - (a - x_0' p - \mu_0' b)(p\lambda_0' + q)(p\lambda_0 + q)\}^2 + pq\,(pd - bq)^2 = 0.$$

This is type XVI.

The two double lines are double generators.

If the two surfaces

$$u = 0, \quad qb - pd = 0$$

intersect in four lines, two of each system, the two lines which cut p, q will be double generators, and that in the same system as p, q will be a double directrix. The surface will then have a fourfold directrix, a double directrix skew to it, and three double generators. The plane $a - x_0' p - \mu_0' q = 0$ can now be written

$$v \equiv \alpha\,(p\lambda_1 + q) + \beta\,(b\lambda_1 + d) = 0,$$

wherein $p\lambda_1 + q$, $b\lambda_1 + d$ is one of the new double generators; the other lies in the plane

$$w \equiv \alpha\,(p\lambda_0' + q) + \beta\,(b\lambda_0' + d) = 0,$$

since

$$\frac{p}{b} = \frac{q}{d} = \frac{p\lambda_1 + q}{b\lambda_1 + q} = \frac{-\beta}{\alpha} = \frac{p\lambda_0' + q}{b\lambda_0' + d}.$$

The first line is a generator, the other the double directrix. If $p\lambda_0' + q \equiv p'$, $p\lambda_1 + q \equiv q'$, and since

$$pd - qb = \frac{1}{\beta\,(\lambda_1 - \lambda_0)}\,[v\,(p\lambda_0' + q) - w\,(p\lambda_1 + q)] \equiv m\,(vp' - wq'),$$

the equation of the surface may be written

$$\{(Ap' + Bq')(vp' - wq') - vp'\,(Cp' + Dq')\}^2 + (vp' - wq')^2(Ep'^2 + Fp'q' + Gq'^2) = 0.$$

The line p', q' is fourfold directrix, v, w double directrix, and q', v; w, p'; $b\lambda_0 + d$, $p'(\lambda_0 + \lambda_1) - q'(\lambda_0 + \lambda_1')$ are double generators. This is type XVII. Several subtypes exist when the pinch-points coincide in various ways.

The two quadrics, u, $pd - qb$, may touch each other along p, q; in this case, $\beta = 0$ in the expression for v, and since $w \equiv \dfrac{p'}{\alpha}$, the equation of the surface becomes

$$[f_1(p, q)(pd - qb) - f_3(p, q)]^2 = pq\,(pd - qb)^2.$$

In this case the surface has a fourfold directrix, a double generator coinciding with it, and three other double generators. This is a limiting case of the last type when the two directrices approach coincidence. It is type XVIII. If the equation be developed, the terms

$$[(\varkappa_0 + \lambda_0 \mu_0')p^2 + 2(\mu_0'(\varkappa_0 + \lambda_0\mu_0') + 1)pq + \mu_0'^2 q^2][pd - qb]^2$$

are infinitesimal of order 4, the other terms are of order 5 or 6 when $p = 0$, $q = 0$. Two planes, $pg + q = 0$, $pg' + q = 0$, are tangent along p, q. Each cuts a fivefold line from the surface. The other two tangent planes are coincident with each other and with the tangent planes of $pd - qb = 0$ along p, q. The surface has self-contact along this line. Every plane through p, q cuts two generators from the surface, which intersect on p, q. If one sheet of the surface be described by continuous motion of each line, the two sheets will pass through each other at the double generators and touch each other along p, q.

Now, consider the case in which the sextic scroll has a double rectilinear directrix. This requires that all of the planes of the form $\alpha\lambda^4 + \dots$ belong to the same axial pencil. The equation can be derived from the general one by writing $a + \varkappa_1 b$, $a + \varkappa_2 b$, $a + \varkappa_3 b$ for c, d, e.

The remaining part of the double curve is of order 9; it has, as in the general case, a fourfold point and lies on a cone of order 5 which has the fourfold point at its vertex. The curve cuts the double directrix in 3 points. Every plane through the double directrix cuts four generators from the surface; these generators intersect in the 6 points of the double curve which lie in the plane, not on the double straight line. This is type XIX. All the generators touch the cone $q^2 - 4pr = 0$, hence the surface is contained in the quadratic congruence defined by the special quadratic complex formed by the tangents to this cone, and the special linear complex whose axis is the line a, b. The two points in which the line a, b cuts the cone are pinch-points or points of intersection of double generators.

If the coordinates of one of these points of intersection cause the t discriminant of the quartic pencil $at^4 + \dots$ to vanish, the generator of the quadric cone which issues from that point will be a double generator of the sextic scroll. The configuration of double lines consists of the double directrix, a double curve of order 8 cutting the directrix in two points and having a triple point and finally a double generator passing through the triple point. This is type XX.

If the coordinates of both points of intersection cause the t discriminant to vanish, both generators issuing from these points will be double generators of the scroll. The double curve is now of order 7, has a double point and cuts the directrix in one point. This is type XXI.

Three double generators cannot appear unless the residual curve consists of a fourfold line; this case was noticed as type XVII. As a subform of XX, the two pinch-points may unite without giving rise to a double generator; i. e. the line a, b may touch the quadric cone.

[III.] $m = 3$, $n = 3$.

By writing

$$\begin{aligned}(as - pd)(pc - ar) - (pb - aq)(sb - qd) &\equiv \psi_1,\\ (rd - cs)(pb - aq) - (as - pd)^2 &\equiv \psi_2,\\ (pd - as)(sb - dq) - (cs - rd)(pc - ar) &\equiv \psi_3,\end{aligned}$$

the equation of the surface is expressed in the form

$$\psi_1 \psi_3 - \psi_2^2 = 0,$$

which has the extraneous factor $pd - as$.

From the identical relation

$$(as - pd)\,\psi_1 + (pc - ar)\,\psi_2 + (aq - bp) \equiv 0,$$

it is seen that ψ_1, ψ_2, ψ_3 pass through the same curve, which is a double curve on the sextic scroll.

ψ_1, ψ_2 intersect in the cubic $pd - as$, $pb - aq$ and the line s, d; they touch along the line a, p.

ψ_2, ψ_3 intersect in the cubic $pd - as$, $cs - rd$ and the line a, p; they touch along the line d, s.

ψ_1, ψ_3 intersect in the quartic $sb - dq$, $pc - ar$ and the two lines p, a; d, s.

The residual curve of order 10 is the double curve of the sextic scroll. The lines a, p; d, s are simple generators of the scroll. This curve has no fourfold points, but has, in general, four triple points. (Salmon's "Algebra," lesson 18.) This scroll is type XXII.

Next, consider the scroll with a triple directrix. Its equation can be derived from the last type by writing $c = a + \varkappa_1 b$, $d = a + \varkappa_2 b$, since all the planes of

the system $at^3 + bt^2 + \ldots = 0$ belong to the same axial pencil. The residual curve is of order 7. Every plane through the triple directrix cuts three generators from the surface which do not in general intersect on the triple line, hence the triple directrix cuts the double curve in four points. This is type XXIII.

The four points in which the line a, b cuts the quartic developable $pt^3 + \ldots = 0$ are pinch-points; when the coordinates of one of these points causes the discriminant of $at^3 + \ldots = 0$ to vanish, the scroll has a double generator which passes through a singular point of the double curve of order 6. This is type XXIV.

In the same manner, the surface may have two or four double generators, or a double and a triple generator. These are types XXV, XXVI, XXVII. The last five types belong to the congruence formed by the tangents to the quartic developable which cut a fixed line. In type XXVI, the double curve of the scroll is of order 3.

From every point of the triple line a, b issue three generators. Let ϖ be a plane containing two generators g, g' issuing from a point of a, b. The plane will cut from the surface a quartic curve having four double points and passing through the intersection with a, b. The curve cannot consist of two conics, for a sextic scroll can have but one, hence the section must consist of straight lines. The one intersecting a, b is a generator and the others cannot be, hence they must all coincide. The double lines consist of two triple directrices, skew to each other, and four double generators. The equation of this surface can be most easily derived from that of type XXIII by putting $r = p + \lambda_1 q$, $s = p + \lambda_2 q$ in that equation.

A large number of subtypes can be derived from this type by having the pinch-points coincide and by the appearance of a triple generator.

Finally, the two skew directrices may coincide. The three generators which issue from each point of a, b lie in a plane passing through a, b. There are four pinch-points on the line a, b. This is type XXVIII. It may have a triple generator, type XXIX.

§3.—*Scrolls Generated by Two Curves.*

Sextic scrolls will now be studied from the dual standpoint, the locus of a line joining corresponding points of two curves.

Let $\xi_i = \xi_i(\lambda)$, $\eta_i = \eta_i(\mu)$, $i = 1, 2, 3, 4$ be the equations of two unicursal curves, the parameters λ, μ being related by the equation

$$f(\lambda, \mu) = 0.$$

Then
$$x_i = \xi_i(\lambda) + \sigma\eta_i(\mu)$$

will define a scroll which is unicursal when the function $f(\lambda, \mu) = 0$ is so. If ξ, η be of order m, n and $f(\lambda, \mu) = 0$ be of degree m_1 in λ, n_1 in μ, ξ will be an n_1-fold curve on the scroll, η will be an m_1-fold line; the scroll will be of order

$$mn_1 + m_1n,$$

but the order will be reduced by unity for every point of intersection of ξ, η, which is a self-corresponding point. When f is unicursal, the parameters λ, μ, σ can be rationally eliminated by a process analogous to that employed for unicursal curves.

A scroll having a simple rectilinear directrix must be unicursal; every plane which contains more than one generator must contain the directrix, and hence $n - 2$ other generators. No simple plane curve of order lower than $n - 1$ can exist on the surface.

The double curve is of order $\frac{1}{2}(n-1)(n-2)$, and it cannot contain double generators as a component part except as one or more generators may coincide with the simple directrix.

When the directrix δ is a simple line on the surface, it cannot be cut by the double curve. The scroll can be generated by joining corresponding points of the line and a unicursal curve c of order $n - 1$ whose plane does not contain δ if it be a plane curve. There are $n - 1$ fundamental types, according as δ passes through a $\varkappa$-fold point on the curve; $\varkappa = 0, 1, \ldots . n - 2$. When δ passes through a $\varkappa$-fold point on c, $\varkappa$ generators coincide with δ and any plane through δ will contain but $n - \varkappa - 1$ generators. When $\varkappa = 1$, the double curve intersects δ in $n - 3$ points; in general, in $\varkappa(n - 2 - \varkappa)$ points. The line δ now counts as a nodal curve of order $\frac{\varkappa}{2}(\varkappa + 1) + r$; the residual curve is of order

$$\tfrac{1}{2}(n-1)(n-2) - \frac{\varkappa}{2}(\varkappa + 1) - r,$$

r being the number of simple coincident tangents of c at points of intersection

with δ. Each generator cuts the residual curve in $n - \varkappa - 2$ points. In particular, if $\varkappa = n - 2$, the $n - 1$-fold line is the only nodal line, and the equation of the scroll is of the form

$$u_n(x, y) + u_{n-1}(x, y)z + v_{n-1}(x, y)w = 0,$$

in which u_{n-1}, v_{n-1} have a common factor of order $n - 2$. In case the curve c has an $n - 2$-fold point and does not cut δ, a special form is one having an $n - 2$-fold line and a simple line

$$xf_{n-1}(x, w) + y\phi_{n-1}(z, w) = 0.$$

In no other case can the nodal curve have a straight line other than δ for a component. These scrolls all belong to a special linear complex; in the two cases just mentioned they are contained in a linear congruence, the former being special, the latter general. The theory will now be applied when $n = 6$.

§4.—*Dual of* (1, 5) *Types.*

When the line δ does not cut the quintic curve, every plane through δ will cut five lines from the surface which intersect in 10 points not lying on δ. They may all coincide, giving type XII, already considered. If the quintic curve does not have a fourfold point, this case is excluded. A double conic, cubic, quartic or quintic is easily proven not to belong to the nodal curve, hence: *When a sextic scroll contains a simple rectilinear directrix which is not a generator, and a quintic curve without a fourfold point, then the nodal curve cannot be reducible.* This c_{10} must accordingly have ∞^1 four-point secants. There are two families according as the nodal curve has a fourfold point or four threefold points. A similar theorem exists for every scroll of even degree $2m$. The nodal curve is of order $(2m - 1)(m - 1)$, and contains ∞^1 secants, each of which cuts it in $2(m - 1)$ points. Curves exist having an infinite family of secants which cut the curve in more than $2(m - 1)$ points, but in that case only one such secant can be drawn through each point of the curve. Type XXX.

When δ cuts the quintic curve c_5 once, the residual curve is of order 9. Type XXXI. The curve may now be composite; its factors are a sextic and a cubic. The scroll may be generated by a (1, 2) correspondence between points on δ and a cubic which cuts δ once, the point of intersection being self-corresponding. The cubic may be twisted, type XXXII.

A different form exists when the cubic is plane. Type XXXIII. A simple

illustration is furnished by joining corresponding points of the line $x = 0, y = 0, z = \mu$ and the curve $x = \lambda^3$, $y = \lambda$, $z = \lambda = \mu(\mu - 1)$.

The residual curve is a sextic which cuts δ twice, and is cut twice by every generator.

The sextic curve may break up into two cubics, so that the nodal curve consists of δ and three (twisted) cubics. The equations may now be written by joining corresponding points of two twisted cubics which intersect in five points; the correspondence being a unicursal (2, 2) such that four points are self-corresponding and the fifth the double element of the (2, 2) correspondence. This is type XXXIV.

No other components of the residual nodal curve can exist.

When δ passes through a node on c_5, δ is a triple line and the residual of order 7. This is type XXXV.

A special form exists when the sections containing a generator cut from the scroll a quintic having a tacnode at the trace of δ; in this case δ counts for a fourfold line (i. e. equivalent to a nodal quartic) and the residue is of order 6. This is type XXXVI.

The sextic may break up into two (twisted) cubics, XXXVII, or finally, a triple conic, as is shown by the scroll

$$(y^3 + x^2z)^2 = x^3yw^2.$$

This is type XXXVIII. The last form is generated by joining corresponding points of $x = 0$, $y = 0$ and $x^2 = y^5$, $z = 0$ by a (1, 1) correspondence.

By applying Clebsch's method* for finding the asymptotic lines of the last scroll, they are found to be

$$x = \lambda^6, \quad y = \lambda^3, \quad z = 20 + c\lambda^4, \quad w = 21\lambda + c\lambda^5.$$

They all have a common osculating plane at the pinch-point (0, 0, 1, 0). Any plane through δ cuts each curve in three points besides the pinch-point on δ, one on each generator in the given plane. Two of the generators are always imaginary. The scroll is contained in the special linear complex $p_{12} = 0$ and in the special quadratic complex $p_{42}^2 + p_{14} \cdot p_{34} = 0$, to which the axis δ belongs.

When δ cuts c_5 in a triple point, the residual is a quartic of the second kind and δ counts for a sextic. This is type XXXIX.

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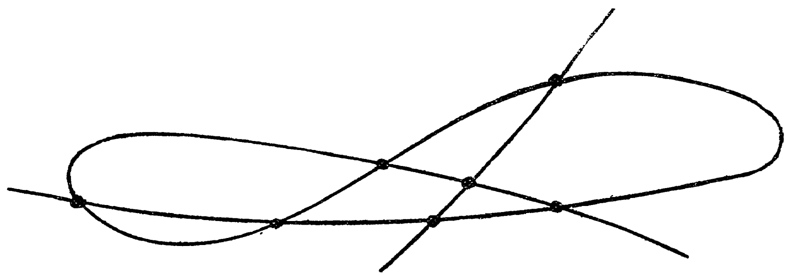
* "Ueber die Haupttangentencurven bei windschiefen Flächen," Crelle, Vol. 68, p. 151.

For particular configurations of the triple point on c_5, the intersection may count for 7, leaving only a nodal cubic. This is type XL.

For example, in the scroll

$$(y^3 - x^2w)^2 = xy\,(xw - xz - y^2)^2,$$

a plane through δ will cut a sextic curve having a singularity of the form



he equations of the nodal cubic are $x = \lambda^3$, $y = \lambda^2$, $z = 1 - \lambda$.

The cubic may become plane, type XLI.

E. g., the scroll

$$(y^3 + x^2z)^2 = x^3yw^2$$

has x, y for a fourfold line and $w = 0$, $y^3 = x^2z$ for a nodal cubic curve. The asymptotic lines are

$$x = 3\lambda^6, \quad y = 3\lambda^4, \quad z = 5 - c\lambda^4\sqrt{\lambda}, \quad w = 8\lambda - c\lambda^5\sqrt{\lambda}.$$

They all have four-point contact at the pinch-point (0, 0, 1, 0).

§5.—*Dual of* (2, 4) *Typ* .

The scrolls, which are the dual of type (2, 4), can be generated by joining corresponding points of a conic section and a unicursal quartic curve. The plane of the conic contains four generators, one through each of the four points in which it cuts the quartic. The conic intersects the nodal curve, which is of order 10, in four points. This is type XLII. The constants may be easily arranged so that the plane of the conic contains a double generator, type XLIII, or two double generators, XLIV, without the residual nodal curve being composite. No other multiple generators can exist, nor any simple plane curve, except the conic, of order less than 4.

In general, no sextic scroll except those contained in a linear congruence

with distinct directrices can have more than two distinct double generators, for, if there be three, the semi-quadric determined by them could intersect the scroll nowhere else ; the complete intersection of the quadric and the scroll is of order 12, hence the residual is made up of lines of the other semi-quadric.

Specializations of the forms of the directrices are:

(α) a double line and a quartic;
(β) a conic and a double conic;
(γ) a conic and a fourfold line ;
(δ) a twofold line and a double conic ;
(ε) a twofold line and a fourfold line, already considered.

(α.) A double line and a unicursal quartic generate a sextic scroll having a double directrix and a residual nodal curve of order 9 ; there may be one or two double generators. These types have already been considered (XIX, XX, XXI).

When δ cuts c_4, it counts for a triple line, and the residual curve is of order 7 ; this is type XLV, it is distinct from XXIII, as the line δ was triple directrix there may be one double generator, type XLVI, or two, type XLVII ; the residual curve must cut δ in three points.

When δ cuts c_4 in a double point, it counts for a nodal curve of order 6, and the residue is a quartic cutting δ three times; type XLVIII. Each generator cuts the quartic once. The configuration of double curves is the same as in XIV, but the types are essentially distinct, as δ is a double generator. There may be a double generator, XLIX. The residue is now a cubic or two double generators and a conic, cutting δ once. Type L.

The last two are distinct types (compare XV, XVI). When δ passes through a triple point on c_4, there can be no other nodal line; δ now counts for a double director and a triple generator. It always belongs to one of the types I to XII.

(β.) A scroll, generated by a (1, 2) correspondence between two conics, has a conic and an octic for nodal curves, LI; there may be one double generator, type LII, or two, type LIII; lines of higher multiplicity may enter. E. g., consider the conics

$$z^2 - xy = 0, \quad w = 0; \quad zw = y^2, \quad x = 0,$$

or

$$x = 0, \quad y = \lambda^2, \quad z = 1, \quad w = \lambda^4,$$
$$x = \lambda^2, \quad y = 1, \quad z = \lambda, \quad w = 0.$$

Then the scroll becomes

$$(w+x)^4 z^2 - 2zwy^2(w+x)^2 + w^2y^4 = (w+x)^3x^2y.$$

The line $w+x=0$, $y=0$ is a fourfold directrix and not a generator. The line $w=0$, $x=0$ is a double line which is cut by any plane in a tacnode of the curve of section. This is type LIV. It differs from XVI by having a tacnodal generator.

(γ.) A scroll may be generated by joining points on a conic to a fourfold line by a (1, 1) correspondence. This type has already been considered. Every scroll having a fourfold line contains either a double line or a conic section. If the line cut the conic, it is a fourfold line and simple generator; there is no other nodal line on the scroll.

(δ). A sextic scroll may be generated by a unicursal (2, 2) correspondence between a straight line and a conic. The residual curve is of order 7, type LV. There may be one double generator and a residual curve of order 6, type LVI. In case the line intersects the conic, it is also a double generator; the residual must consist of two double generators, as four intersections of an arbitrary generator and the nodal curve are already accounted for. This type was found before (XVI).

§6.—*Scrolls having Two Double Conics.*

When two conics, which lie in different planes but which have two points of intersection, are put in (2, 2) correspondence in such a way that each point of intersection is a single self-corresponding point, the lines joining corresponding points will be a sextic scroll. Let the conics be

$$\begin{aligned} &x=\lambda, \quad y=\lambda^2, \quad z=0, \quad w=1, \\ &x=0, \quad y=\mu^2, \quad z=\mu, \quad w=1. \end{aligned}$$

The general form of the correspondence is

$$a\lambda^2\mu + b\lambda\mu^2 + c\lambda^2 + d\lambda\mu + e\mu^2 + f\lambda + g\mu = 0.$$

The equation of the scroll can be written in the form

$$\begin{vmatrix} bz-ax-cw & (e+c)z-by-dx & f & ey+gx & 0 \\ 0 & bz-ax & c & by+fw+dx-ez & ey+gx-fz \\ wz & x^2-z^2 & y & 0 & 0 \\ 0 & w & 1 & -z & 0 \\ 0 & 0 & z & x^2-wy & zy \end{vmatrix} = 0.$$

The residual curve is of order 5 and is cut by every generator in two points. When the surface is unicursal, a double generator exists. The latter is expressed by a double point not at (0, 0) nor at (∞, ∞) on the cubic curve in λ, μ. The plane $z = 0$ contains the two generators $ax + cw = 0$, $ey + gx = 0$. The plane $x = 0$ contains the generators $cy + fz = 0$, $ew + bz = 0$. This is type LVII.

When
$$\frac{f}{g} = \frac{b}{a} \text{ and } c = e$$
there is a third double conic lying in the plane $gx - fz = 0$ and passing through the points (0, 0, 0, 1), (0, 1, 0, 0). The residual is now a twisted cubic which meets each generator once. Type LVIII. If the new conic touches the plane $y = 0$ or $w = 0$, the scroll reduces to a quintic and the common tangent plane.

If
$$c + e = 0, \quad f = 0, \quad b + ad = 0,$$
the conic is replaced by a double rectilinear directrix and a double generator. There is no other double generator. The equations of the directrix are
$$ax = a^2y + (e + ag)\,w, \quad ax + adz = ew.$$
The residual is a twisted quartic which cuts every generator once. Type LIX. A particular case of these types is when the two conics touch each other; the surface mentioned by de la Gournerie* as a factor of a composite form of the "quadrispinale" of order 8 is the result.

If $c = e = 0$, the double generator is the line joining the points of intersection of the two conics. If, in this case, $\dfrac{f}{g} = \dfrac{b}{a}$, the third double conic breaks up into the double generator and the double rectilinear directrix
$$bdz + y + bgw = 0, \quad gx = fz.$$

If a sextic scroll has three double conics belonging to an axial pencil, the common chord cannot be a double generator.

When, instead of each point of intersection being a simple self-corresponding point, one of them is a double element, the surface can have no double generators, nor can the surface have a rectilinear directrix. A third conic cannot lie in any plane passing through the points of intersection of the first two. Through the point (0, 1, 0, 0) pass four generators.

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* "Recherches sur les surfaces réglées." Paris, 1867. See p. 156.

If the conics have but one point of intersection, their equations may be written

$$x = \mu, \quad y = \mu^2, \quad z = 0, \quad w = 1,$$
$$x = 0, \quad y = \lambda^2, \quad z = \lambda, \quad w = 1 - \varkappa\lambda^2,$$

and the equation of the correspondence is

$$a\lambda^2\mu^2 + b\lambda^2\mu + c\lambda\mu^2 + d\lambda^2 + e\lambda\mu + f\mu^2 = 0.$$

The residual curve is of order 6 and cuts each generator twice. Type LX. The surface belongs to a complex if $d + f = 0$ and the complex becomes special, if in addition, $e = 0$. The axis of the complex is defined by

$$y = 0, \quad bx - cz + f = 0.$$

The residual curve is of order 5 and cuts each generator once. Type LXI.

When $\varkappa = 0$ and a correspondence of the form

$$a\lambda^2\mu + b\lambda^2 + c\lambda\mu^2 + d\lambda\mu + e\mu^3 + f\mu^2 = 0$$

exists, the conic in $x = 0$ is a triple conic and the other is a double conic. The residual curve is also a conic. It has one point in common with each of the given conics, hence the surface belongs to the unicursal (2, 2) family with one double element. When $d = 0$ and $f + b = 0$, the third conic lies in the common tangent plane of the two given conics at a point of intersection, and touches the double conic. Type LXII.

§7.—*Dual of* (3, 3) *Types.*

Scrolls of the dual of the (3, 3) type may be generated by joining corresponding points of two twisted cubics or unicursal plane cubics by a (1, 1) correspondence, making three varieties. In case the scroll contains two plane cubics, it counts as a separate type, LXIII. No curve of order less than three can be a simple directrix of the scroll.

A double generator may exist, type LXIV, a triple generator, type LXV. The scrolls may also be generated by joining corresponding points of a cubic and a unicursal quartic which intersects it in a self-corresponding point, and similarly for curves of order 5 or 6.

One cubic curve may be replaced by a triple line, but this form has already been considered; every scroll containing a triple line must also contain a cubic curve or another triple line. When the line meets the cubic, it becomes a four-

fold line, and the residual curve is of order 4. It is a distinct type, LXVI. A double generator may be present, type LXVII, or two; the residual curve becoming a double conic, LXVIII.

§8.—*Sextic Developables.*

The forms of sextic developables are classified in the doctor dissertation of Professor Schwarz* according to characteristics of cuspidal edge and double curve. The five possible types are discussed in Salmon's Geometry, pp. 314–318. The general case is a specialization of LX and its dual; two forms appear from the latter when one or two inflexional tangents are present. They are most easily studied by the methods employed in types XXII to XXIX. The developables of order 6 are all unicursal (planar). They will not be included in the following table.

§9.—*Table of Forms of Unicursal Sextic Scrolls.*

By writing xc_n^m for x distinct curves of order n, each counting as an m-fold curve on the scroll, g^l as an l-fold generator, any line-symbol with a bar over it for coincident tangent planes and $[c_3]$ for a plane cubic, the characteristics of the scrolls obtained may be expressed as follows:

I. c_1^5,
II. $(c_1^4 + g^2)$,
III. $(c_1^3 + 2g^2)$,
IV. $(c_1^2 + g^3)$,
V. $(c_1^1 + g^4)$,
VI. $(c_1^3 + 2\bar{g}^2)$,
VII. $(c_1^3 + g' + g_1^2)$,
VIII. $(c_1^2 + \bar{g}^3)$,
IX. $(c_1^1 + g + \bar{g}_1^3)$,
X. $(c_1^1 + \bar{g}^2 + \dot{g}_1^2)$,
XI. $(c_1^1 + \bar{g}^{\,4})$,
XII. $(c_1^1 + c_1^5)$,
XIII. $c_{10,\,4}^2$,
XIV. $c_1^4 + c_4^2$,
XV. $c_1^4 + c_3^2 + g^2$,
XVI. $c_1^4 + c_2^2 + 2g^2$,
XVII. $c_1^4 + c_1^2 + 3g^2$,
XVIII. $c_1^4 \equiv c_1'^2 + 3g^2$,
XIX. $c_1^2 + c_{9,\,4}^2$,
XX. $c_1^2 + c_{8,\,3}^2 + g^2$,
XXI. $c_1^2 + c_{7,\,2}^2 + 2g^2$,
XXII. $c_{10,\,3}^2$,
XXIII. $c_1^3 + c_7^2$,
XXIV. $c_1^3 + c_6^2 + g^2$,
XXV. $c_1^3 + c_5^2 + 2g^2$,
XXVI. $3c_1^3 + 4g^2$,

* "De superficiebus in planum explicabilibus primorum septem ordinum." Crelle, Vol. 64.

XXVII. $2c_1^3 + g^3 + g'^2$,
XXVIII. $c_1^3 \equiv c_1'^3 + 4g^2$,
XXIX. $c_1^3 \equiv c_1'^3 + g^3 + g'^2$,
XXX. $c_{10}^2 + c_1^1$,
XXXI. $(c_1^1 + g) + c_9^2$.
XXXII. $(c_1^1 + g) + c_6^2 + c_3^2$,
XXXIII. $(c_1^1 + g) + c_6^2 + [c_3^2]$,
XXXIV. $(c_1^1 + g) + 3c^2$,
XXXV. $(c_1^1 + g^2) + c_7^2$,
XXXVI. $(\overline{c_1^1 + g^2)} + c_6^2$,
XXXVII. $(c_1^1 + g^2) + 2c_3^2$,
XXXVIII. $(c_1^1 + g^2) + c_3^3$,
XXXIX. $(c_1^1 + g^3) + c_4^3$,
XL. $\overline{(c_1^1 + g^3)} + c_3^2$,
XLI. $(c_1^1 + g^3) + [c_3^2]$,
XLII. $c_{10}^2 + c_2^1$,
XLIII. $c_9^2 + c_2^1 + g^2$,
XLIV. $c_8^2 + c_2^1 + 2g^2$,
XLV. $(c_1^2 + g) + c_7^2$,
XLVI. $(c_1^2 + g) + c_6^2 + g^2$,
XLVII. $(c_1^2 + g) + c_5^2 + 2g^2$,
XLVIII. $(c_1^2 + g^2) + c_4^2$,
XLIX. $(c_1^2 + g^2) + c_3^2 + g^2$,
L. $(c_1^2 + g^2) + c_2^2 + 2g^2$,
LI. $c_8^2 + c_2^2$,
LII. $c_7^2 + c_2^2 + g^2$,
LIII. $c_6^2 + c_2^2 + 2g^2$,
LIV. $c_1^4 + c_2^2 + \overline{2g^2}$,
LV. $c_1^2 + c_2^2 + c_7^2$,
LVI. $c_1^2 + c_2^2 + c_6^2 + g^2$,
LVII. $2c_2^2 + c_5^2 + g^2$,
LVIII. $3c_2^2 + c_3^2 + g^2$,
LIX. $2c_2^2 + c_4^2 + g^2 + c_1^2$,
LX. $2c_2^2 + c_6^2$,
LXI. $2c_2^2 + c_5^2 + c_1^2$,
LXII. $2c_2^2 + c_2^3$,
LXIII. $c_{10}^2 + [2c_3^1]$,
LXIV. $c_9^2 + 2c_3^1 + g^2$,
LXV. $c_7^2 + g^3 + [2c_3^1]$,
LXVI. $(c_1^3 + g^2) + c_4^2$,
LXVII. $(c_1^3 + g^2) + c_3^2 + g^2$,
LXVIII. $(c_1^3 + g^2) + c_2^2 + 2g^2$.

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